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The $\mathcal{N} = 1$ supersymmetric Landau problem and its supersymmetric Landau level projections: the $\mathcal{N} = 1$ supersymmetric Moyal–Voros superplane

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Abstract
By considering Landau level projections of the $\mathcal{N} = 1$ supersymmetrized Landau problem remaining nontrivial under $\mathcal{N} = 1$ supersymmetry transformations, the algebraic structures of the $\mathcal{N} = 1$ supersymmetric covariant non(anti)commutative superplane analogue of the ordinary $\mathcal{N} = 0$ noncommutative Moyal–Voros plane are identified. In contradistinction to all discussions available in the literature on the supersymmetrized Landau problem, the relevant Landau level projections do not amount to taking a massless limit of the Landau system.

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1. Introduction

Over the years, the classic textbook example [1] of the quantum Landau problem has remained a constant source of fascination and inspiration, in fields apparently so diverse as condensed matter physics, the fundamental unification of gravity with high-energy quantum physics or purely mathematical studies in noncommutative deformations of geometry. The quantum Landau system with its Landau level structure of energy quantum states provides a natural model for the integer and fractional quantum Hall effects, whether in its commutative

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or noncommutative formulations [2, 3]. The noncommutative geometry resulting from a projection onto any of the Landau levels [4, 5] provides the basic example of deformation quantization through the Moyal–Weyl (or Voros) $*$-product [6]. The same algebraic structures are also realized in M-theory for specific background field configurations [7].

Accounting for the spin degrees of freedom of the charged particle in the Landau problem—after all this is the physical situation for spin-1/2 electrons—raises the possibility of extending further such algebraic structures in a manner consistent with supersymmetry. Through appropriate projections onto Landau levels, one wonders then how the noncommutative Euclidean Moyal–Voros plane (or torus) extends into a Grassmann graded noncommutative variety, whose originally commuting and anticommuting variables now possess some deformed algebra representative of a non(anti)commutative Grassmann graded geometry with supersymmetry.

Extensions of the Landau problem including Grassmann graded degrees of freedom and algebraic structures of the supersymmetric type have been considered over the years, whether in a planar, toroidal or spherical geometry [8], but apparently never explicitly with the above purpose in mind. Deformations of the algebraic structures of even a Grassmann graded Landau problem have so far not led to a projection onto Landau levels on which supersymmetry acts nontrivially [9, 10]. We also note that the supersymmetry inherent to the Landau problem with a spin-1/2 charged particle of gyromagnetic factor $g = 2$ does not survive a noncommutative geometry on the Euclidean plane [11]. More specifically, all previous approaches, whether those of [8] (and references therein) or even [12], have considered a massless limit of the Landau problem or some procedure equivalent to that limit. In contradistinction, in the present work such a limit is not considered at all, but rather the guiding principle is the requirement of preserving a nontrivial realization of supersymmetry on the projected sector of the supersymmetric Landau system.

With the above condensed matter and mathematical physics contexts in the back of one's mind as motivations for the potential relevance of such an analysis, in this work we present a first investigation addressing the question raised above, restricted to the simplest case of a single supersymmetry, $\mathcal{N} = 1$. Results for a larger number of supersymmetries are to be discussed elsewhere [13]. Naively one might expect that besides the noncommutative coordinates of the Euclidean Moyal–Voros plane, the nonanticommutative Grassmann odd sector may amount simply to some rescaling of the associated Clifford algebra, and retain its commutativity with the Grassmann even sector. Surprisingly perhaps, this is not what our analysis reveals. Rather, in what may as well be called the $\mathcal{N} = 1$ supersymmetric Moyal–Voros superplane, there is a transmutation of sorts, between the fermionic and some of the bosonic degrees of freedom, resulting in degrees of freedom of fermionic character and yet possessing an integer spin. The spin-1/2 degrees of freedom end up coinciding with one of the chiral modes of the bosonic sector, and vice versa, such that the latter obeys a fermionic statistics nonetheless. Finally these bosonic–fermionic coordinates solder two copies of the ordinary $\mathcal{N} = 0$ Moyal–Voros plane, to produce its $\mathcal{N} = 1$ superplane extension.

In section 2 we review the results of the ordinary and $\mathcal{N} = 1$ supersymmetric Landau problem relevant to our analysis. The $\mathcal{N} = 1$ supersymmetric Landau problem is solved explicitly for a particle moving in the plane with a static and homogeneous magnetic field perpendicular to the plane. Section 3 then considers projections onto Landau levels preserving nontrivially the $\mathcal{N} = 1$ supersymmetry, and identifies the algebraic structures resulting from such projections which characterize the $\mathcal{N} = 1$ supersymmetric Moyal–Voros superplane, with in particular the boson–fermion transmutation mentioned above. Section 4 presents the final and complete form of the $\mathcal{N} = 1$ supersymmetric Moyal–Voros superplane, as well as some concluding remarks.

2. The ordinary and $\mathcal{N}$ = 1 supersymmetric Landau problem

To set the context and notation of our analysis we briefly recall the ordinary Landau problem of a charged particle of mass m moving in the Euclidean plane of Cartesian coordinates (x_1, x_2) and subjected to a static and homogeneous magnetic field perpendicular to that plane. In the symmetric gauge the system is defined by the following Lagrangian:

$$L_0 = \tfrac{1}{2}m\left(\dot{x}_1^2 + \dot{x}_2^2\right) - \tfrac{1}{2}B\left(\dot{x}_1 x_2 - \dot{x}_2 x_1\right). \tag{1}$$

In this expression, B stands for the magnetic field with a normalization which includes the charge of the particle. Furthermore, without loss of generality it is assumed that the (right-hand) orientation of the plane (x_1, x_2) is such that $B > 0$ (as usual a dot above a quantity stands for its first-order time derivative).

The corresponding Hamiltonian of the system, p_i being the conjugate momentum to x_i $(i = 1, 2)$, is obtained as

$$\begin{aligned} H_0 = \dot{x}_1 p_1 + \dot{x}_2 p_2 - L_0 &= \frac{1}{2m}\left(p_1 + \frac{1}{2}Bx_2\right)^2 + \frac{1}{2m}\left(p_2 - \frac{1}{2}Bx_1\right)^2 \\ &= \frac{1}{2m}\left(p_1^2 + p_2^2\right) + \frac{1}{2}m\left(\frac{\omega_c}{2}\right)^2\left(x_1^2 + x_2^2\right) - \frac{1}{2}\omega_c(x_1 p_2 - x_2 p_1), \end{aligned} \tag{2}$$

where $\omega_c = B/m$ denotes the cyclotron angular frequency. One reason for the above choice of the circular gauge is that it makes manifest the invariance of the dynamics under $SO(2)$ rotations in the plane, generated by

$$L \equiv L_{\text{Noether}} = x_1 p_2 - x_2 p_1, \tag{3}$$

which, besides the total energy H_0, is a second and independent constant of the motion.

Introducing the standard chiral or helicity Fock algebra operators[6] satisfying

$$[L, a_\pm] = \mp\hbar a_\pm, \qquad \left[L, a_\pm^\dagger\right] = \pm\hbar a_\pm^\dagger, \tag{4}$$

i.e. the creation operator $a_+^\dagger$ (resp., $a_-^\dagger$) creates a quantum carrying a unit $(+\hbar)$ (resp., $(-\hbar)$) of angular momentum, the Hamiltonian and angular momentum can be written as

$$H_0 = \hbar\omega_c a_-^\dagger a_- + \tfrac{1}{2}\hbar\omega_c, \qquad L = \hbar\left(a_+^\dagger a_+ - a_-^\dagger a_-\right). \tag{5}$$

An orthonormal chiral Fock state basis, denoted by $|n_+, n_-\rangle$, which diagonalizes both H_0 and L, can now be built in the standard way by acting with these creation operators on the normalized Fock vacuum state $|\Omega\rangle \equiv |0, 0\rangle$.

Let us now consider a fixed Landau sector at level $N = 0, 1, 2, \ldots$ and the associated projection operator

$$\mathbb{P}_N = \sum_{n_+=0}^{\infty} |n_+, N\rangle\langle n_+, N|, \qquad \mathbb{P}_N^\dagger = \mathbb{P}_N, \qquad \mathbb{P}_N^2 = \mathbb{P}_N. \tag{6}$$

Given any operator A, we denote by $\overline{A}$ the associated operator projected onto the Landau sector at level N, $\overline{A} = \mathbb{P}_N\, A\, \mathbb{P}_N$. In particular, note that $\overline{a}_- = 0$, $\overline{a_-^\dagger} = \overline{a}_-^\dagger = 0$.

As is well known, upon projection of the coordinates and conjugate momenta one conjugate pair vanishes identically[7], or, equivalently, projection onto any given Landau level

[6] Namely $a_\pm = (a_1 \mp \mathrm{i}a_2)/\sqrt{2}$, with $x_i = \sqrt{\hbar/B}(a_i + a_i^\dagger)$ and $p_i = -\mathrm{i}\sqrt{\hbar B}(a_i - a_i^\dagger)/2$.

[7] The pair which vanishes is precisely the one that contributes in the Hamiltonian H_0. This fortuitous circumstance is such that projection onto the Landau level N is equivalent to taking a massless limit of the initial system, $m \to 0^+$, provided the quantum energy $\hbar\omega_c(N + 1/2)$ of that Landau level is first subtracted from the Hamiltonian in order that all Landau levels decouple but for the one under consideration.

implies that the conjugate momentum operators p_i are no longer independent of the Cartesian coordinate operators x_i, and become proportional to these. As a consequence, the latter no longer commute as they do when acting on the set of all Landau levels. Rather, as is well known, on the Landau level N one now finds

$$[\overline{x}_1, \overline{x}_2] = -\mathrm{i}\frac{\hbar}{B}\mathbb{P}_N. \tag{7}$$

Through projection onto the Landau level N, the four-dimensional phase space of the system has been projected to a two-dimensional phase space with as conjugate pair of variables the two (projected) Cartesian coordinates of the plane obeying, up to some normalization factor, the usual Heisenberg algebra. The original commuting algebra spanned by the operators x_1 and x_2 is deformed into a noncommutative algebra spanned by the operators $\overline{x}_1$ and $\overline{x}_2$. These latter two operators define the noncommutative Moyal–Voros plane characterized by the nonvanishing commutator (7). From the present point of view, the noncommutative Moyal–Voros plane is seen to be nothing else than the representation space of a single Fock algebra, in the present instance that of the right-handed chiral Fock algebra restricted to the Landau level N. In effect, this algebra also corresponds to the noncommutative quantum phase space of a one-degree-of-freedom system obeying the Heisenberg algebra.

One can also envisage the possibility of projecting the quantum Landau problem onto a collection of $(M+1)$ successive Landau levels ($M = 0, 1, 2, \ldots$), in terms of a projector:

$$\mathbb{P}_{N,M} = \mathbb{P}_N + \mathbb{P}_{N+1} + \cdots + \mathbb{P}_{N+M}; \tag{8}$$

the value $M = 0$ corresponding to the discussion above. However, such a procedure does not appear to offer any particular advantage (in the absence of supersymmetry at this stage), and we shall not consider it any further here.

Finally let us point out that the above construction involving projection onto any given Landau sector of the full Hilbert space as such is in no way related to the choice of Hamiltonian. Rather, given the parameter B in combination with $\hbar$, out of the (x_i, p_i) Heisenberg algebra it is always possible to define the chiral Fock algebras for $\left(a_\pm, a_\pm^\dagger\right)$, hence the Landau sectors $|n_+, N\rangle$ for a fixed $N = 0, 1, 2, \ldots$. It only so happens that the chiral Fock states $|n_+, n_-\rangle$ also diagonalize the operators H_0 and L. Had the Hamiltonian operator been different from H_0 (for instance by adding an interaction potential energy to it, $V(x_1, x_2)$), its eigenspectrum would not have coincided with the chiral Fock state basis $|n_+, n_-\rangle$. Still, independently from this one may consider the different projections onto Landau sectors discussed above.

The action for the $\mathcal{N} = 1$ supersymmetric nonrelativistic particle coupled to an arbitrary background magnetic field in the d-dimensional Euclidean space is given by [14]

$$\begin{aligned} S_1 &= \int \mathrm{d}t\,\mathrm{d}\theta \left\{ -\frac{1}{2} m D^2 X_i D X_i + \mathrm{i} D X_i A_i(X_i) \right\} \\ &= \int \mathrm{d}t \left\{ \frac{1}{2} m \dot{x}_i^2 - \frac{1}{2}\mathrm{i} m \dot{\lambda}_i \lambda_i + \dot{x}_i A_i(x_i) - \frac{1}{2}\mathrm{i} B_{ij}(x_i)\lambda_i\lambda_j \right\}. \end{aligned} \tag{9}$$

Here the θ and $\lambda_i(t)$ are real Grassmann variables. In particular the $\lambda_i(t)$ correspond to the real-valued spin degrees of freedom of the particle. D is the supercovariant derivative

$$D = \partial_\theta - \mathrm{i}\theta\partial_t, \qquad D^\dagger = D, \qquad D^2 = -\mathrm{i}\partial_t. \tag{10}$$

The $X_i(t,\theta)$, $X_i^*(t,\theta) = X_i(t,\theta)$ are the real Grassmann even supercoordinates

$$X_i(t,\theta) = x_i(t) + \mathrm{i}\theta\lambda_i(t), \qquad i = 1, 2, \ldots, d. \tag{11}$$

The variables $x_i(t)$ are Grassmann even, hence bosonic variables, corresponding to the real valued coordinates of the particle in the Euclidean space. Note that as θ is a scalar under

Euclidean space transformations and in particular rotations, both x_i and λ_i transform as $SO(d)$ vectors under spatial rotations.

By construction, this action is invariant under the infinitesimal supersymmetry transformations

$$\delta_\epsilon x_i(t) = \epsilon\lambda_i(t), \qquad \delta_\epsilon \lambda_i(t) = \mathrm{i}\epsilon\dot{x}_i(t), \tag{12}$$

with ϵ a real Grassmann odd constant infinitesimal parameter. What is noteworthy about this system is that $\mathcal{N} = 1$ supersymmetry precludes any possible interaction potential energy besides the magnetic coupling. Thus, for instance, an electrostatic or electric coupling of such a charged particle is incompatible with $\mathcal{N} = 1$ supersymmetry.

The conserved Noether generator for supersymmetry transformations is readily found to be given by

$$Q_{\text{Noether}} = \mathrm{i}m\lambda_i\dot{x}_i. \tag{13}$$

Unless the background magnetic field displays specific symmetry properties, the system does not possess any further conserved quantities besides Q_{Noether} and its total energy, namely its canonical Hamiltonian.

The Hamiltonian formulation of the system is provided by the conjugate pairs of Grassmann even and Grassmann odd phase space variables. The bosonic coordinates, x_i, possess Grassmann even conjugate momenta, $p_i = m\dot{x}_i + A_i(x_i)$, with the canonical Poisson brackets

$$\{x_i, p_j\} = \delta_{ij}. \tag{14}$$

The action being already of first order in $\dot{\lambda}_i$, the fermionic variables λ_i are conjugate to themselves, with the following Grassmann graded Poisson brackets[8]:

$$\{\lambda_i, \lambda_j\} = -\mathrm{i}\frac{1}{m}\delta_{ij}. \tag{15}$$

The canonical Hamiltonian then reads

$$H = \frac{1}{2m}(p_i - A_i(x_i))^2 + \frac{1}{2}\mathrm{i}B_{ij}\lambda_i\lambda_j, \tag{16}$$

while for the Noether supercharge

$$Q_{\text{Noether}} = \mathrm{i}\lambda_i(p_i - A_i(x_i)). \tag{17}$$

Quantization of the system is straightforward enough. Upon rescaling of the fermionic operators an anticommuting Clifford algebra is obtained in the Grassmann odd sector, while the Grassmann even one obeys the usual Heisenberg algebra of Euclidean space. Namely, writing

$$\lambda_i = \sqrt{\frac{\hbar}{2m}}\gamma_i, \qquad \lambda_i^\dagger = \lambda_i, \qquad \gamma_i^\dagger = \gamma_i, \tag{18}$$

the quantum system is defined by the algebra of the following (only nonvanishing) commutation and anticommutation relations of Hermitian operators:

$$[x_i, p_j] = \mathrm{i}\hbar\delta_{ij}\mathbb{I}, \qquad \{\gamma_i, \gamma_j\} = 2\delta_{ij}\mathbb{I}, \tag{19}$$

while one has

$$H = \frac{1}{2m}(p_i - A_i(x_i))^2 + \frac{\hbar}{m}B_{ij}(x_i)\sigma_{ij}, \qquad Q_{\text{Noether}} = \mathrm{i}\sqrt{\frac{\hbar}{2m}}\gamma_i(p_i - A_i(x_i)), \tag{20}$$

[8] Applying Dirac's analysis of constraints, these brackets are the Dirac brackets resulting from fermionic second-class constraints [15].

where

$$\sigma_{ij} = \tfrac{1}{2}\mathrm{i}[\gamma_i, \gamma_j] \tag{21}$$

generate the spinor representation of $SO(d)$. Hence, indeed, the system describes a spin-1/2 charged point particle coupled to the magnetic field, possessing a total energy which includes the magnetic spin coupling energy for a gyromagnetic factor of $g = 2$. This very specific value for this coupling is a direct consequence of supersymmetry, as is well known.

An explicit analytic solution of the quantum system is possible only for specific magnetic field configurations. A constant field is certainly such a case, with the vector potential then linear in the coordinates for a particular class of gauges. Through an appropriate rotation in the Euclidean space it is always possible to bring the magnetic field B_{ij} to lie only in the (12) plane, say, in which case the motion of the particle in all directions perpendicular to that plane is free, any additional potential being forbidden by the $\mathcal{N} = 1$ supersymmetry. Henceforth, we restrict the discussion to the $\mathcal{N} = 1$ supersymmetric Landau problem in the plane.

In the above setup the magnetic field B_{12} corresponds to that of the ordinary Landau problem discussed earlier, $B = B_{12}$. Working once again in the circular gauge for the vector potential, the action of the system is manifestly invariant under $SO(2)$ rotations in the plane, $i, j = 1, 2$. The Noether generator for such infinitesimal rotations is found to be

$$L_{\text{Noether}} = \epsilon_{ij} x_i (m\dot{x}_j + A_j(x_i)) - \tfrac{1}{2}\mathrm{i}m\epsilon_{ij}\lambda_i\lambda_j = x_1 p_2 - x_2 p_1 - \tfrac{1}{2}\mathrm{i}m[\lambda_1, \lambda_2], \tag{22}$$

ϵ_{ij} being the antisymmetric symbol with $\epsilon_{12} = +1$. This quantity thus defines the total angular momentum of the system, inclusive of the particle spin contribution.

Hence, given this magnetic field configuration, the system possesses three conserved quantities, of which all the Poisson brackets vanish on account of their invariance under supersymmetry and translations in time, namely H, Q_{Noether} and L_{Noether}. At the quantum level, these three operators thus commute and may be diagonalized in a common basis of eigenstates.

Note also that the Grassmann graded phase space of the system consists of four bosonic real variables, (x_i, p_i), and two fermionic real variables, λ_i. Phase space is thus of dimension (4|2) in that sense.

In the present case $d = 2$ and the operators $\gamma_i = \lambda_i\sqrt{2m/\hbar}$ define the $SO(2)$ Clifford algebra, $\{\gamma_i, \gamma_j\} = 2\delta_{ij}\mathbb{I}$. A possible representation[9] of the fermionic sector of the system is thus given by the Pauli matrices σ_α ($\alpha = 1, 2, 3$) as follows:

$$\gamma_1 = \sigma_1, \qquad \gamma_2 = \sigma_2, \qquad [\gamma_1, \gamma_2] = 2\mathrm{i}\sigma_3. \tag{23}$$

Consequently one has

$$H = \frac{1}{2m}\left(p_1 + \frac{1}{2}Bx_2\right)^2 + \frac{1}{2m}\left(p_2 - \frac{1}{2}Bx_1\right)^2 - \frac{\hbar}{2m}B\sigma_3, \tag{24}$$

$$L_{\text{Noether}} = x_1 p_2 - x_2 p_1 + \frac{1}{2}\hbar\sigma_3, \tag{25}$$

$$Q_{\text{Noether}} = \mathrm{i}\sqrt{\frac{\hbar}{2m}}\left[\sigma_1\left(p_1 + \frac{1}{2}Bx_2\right) + \sigma_2\left(p_2 - \frac{1}{2}Bx_1\right)\right], \tag{26}$$

making it explicit that indeed the particle is of spin 1/2, with an energy which is decreased when the spin projection onto the axis perpendicular to the plane is aligned with the magnetic field, as it should. Spin-up and -down states differ in energy by the quantum of energy $\hbar\omega_c$,

[9] Any other representation is unitarily equivalent to the present one by some $SU(2)$ transformation acting on the Pauli matrices.

which is also the quantum of energy excitations in the bosonic sector. This equality of energy quanta values in both sectors is required by supersymmetry, and is intimately related to the value $g = 2$ for the gyromagnetic factor of the charged particle. Incidentally, this equality of bosonic and fermionic energy gaps confirms once again that an $\mathcal{N} = 1$ supersymmetry forbids any extra potential, which would otherwise break the equal spacing in energy of Landau levels and lift their degeneracies in an *a priori* arbitrary fashion in the bosonic sector.

Using in the bosonic sector the chiral Fock algebras described earlier and introducing the following chiral combinations in the fermionic spin-1/2 sector as well:

$$\sigma_\pm = \tfrac{1}{2}\left(\sigma_1 \pm \mathrm{i}\sigma_2\right), \qquad \sigma_\pm^\dagger = \sigma_\mp, \qquad \sigma_1 = \sigma_+ + \sigma_-, \qquad \sigma_2 = -\mathrm{i}(\sigma_+ - \sigma_-), \tag{27}$$

the algebra of the elementary degrees of freedom of the system is

$$\left[a_\pm, a_\pm^\dagger\right] = \mathbb{I}, \qquad \{\sigma_+, \sigma_-\} = \mathbb{I}, \qquad \sigma_+^2 = 0, \qquad \sigma_-^2 = 0. \tag{28}$$

Furthermore, one also has

$$\{\sigma_3, \sigma_\pm\} = \pm 2\sigma_\pm, \qquad [\sigma_+, \sigma_-] = \sigma_3, \qquad [\sigma_3, \sigma_\pm] = \pm 2\sigma_\pm. \tag{29}$$

By direct substitution one finds

$$H = \hbar\omega_c\left(a_-^\dagger a_- + \frac{1}{2}\right) - \frac{1}{2}\hbar\omega_c\sigma_3 = \hbar\omega_c\left(a_-^\dagger a_- + \frac{1-\sigma_3}{2}\right), \tag{30}$$

$$L \equiv L_{\text{Noether}} = \hbar\left(a_+^\dagger a_+ - a_-^\dagger a_- + \frac{1}{2}\sigma_3\right), \tag{31}$$

$$Q_0 = \frac{\mathrm{i}}{\sqrt{\hbar}} Q_{\text{Noether}} = \mathrm{i}\sqrt{\hbar\omega_c}\left(\sigma_- a_- - \sigma_+ a_-^\dagger\right). \tag{32}$$

The normalization and phase of Q_0 are chosen such that

$$Q_0^2 = H, \qquad Q_0^\dagger = Q_0. \tag{33}$$

The action of these operators on the degrees of freedom is

$$\left[H, a_+^{(\dagger)}\right] = 0, \qquad \left[H, a_-^\dagger\right] = \hbar\omega_c a_-^\dagger, \qquad [H, a_-] = -\hbar\omega_c a_-, \qquad [H, \sigma_\pm] = \mp\hbar\omega_c\sigma_\pm, \tag{34}$$

$$\left[L, a_\pm^\dagger\right] = \pm\hbar\, a_\pm^\dagger, \qquad [L, a_\pm] = \mp\hbar\, a_\pm, \qquad [L, \sigma_\pm] = \pm\hbar\,\sigma_\pm, \tag{35}$$

$$\left[Q_0, a_+^{(\dagger)}\right] = 0, \qquad \left[Q_0, a_-^\dagger\right] = \mathrm{i}\sqrt{\hbar\omega_c}\,\sigma_-, \qquad [Q_0, a_-] = \mathrm{i}\sqrt{\hbar\omega_c}\,\sigma_+, \tag{36}$$

$$\{Q_0, \sigma_+\} = \mathrm{i}\sqrt{\hbar\omega_c}\,a_-, \qquad \{Q_0, \sigma_-\} = -\mathrm{i}\sqrt{\hbar\omega_c}\,a_-^\dagger. \tag{37}$$

It may easily be checked that all commutators of the three operators H, L and Q_0 do indeed vanish. Note that $\mathcal{N} = 1$ supersymmetry transformations mix only the left-handed chiral bosonic mode with the fermionic degrees of freedom, leaving the right-handed chiral bosonic mode invariant. This simplest form of supersymmetry thus has a 'chiral preference' in the presence of the magnetic field which distinguishes these two chiralities and breaks time reversal invariance, in spite of the fact that supersymmetry transformations treat spin-up and -down states both on an equal footing. Incidentally, this property of $\mathcal{N} = 1$ supersymmetry transformations implies that the specific combinations $(p_1 + Bx_2/2)$ and $(p_2 - Bx_1/2)$ are supersymmetry invariants.

A construction of a basis of states for the quantized system, which furthermore diagonalizes these operators, is obvious. In the bosonic sector one has the chiral Fock

states $|n_+, n_-\rangle$ while in the fermionic sector one has the orthonormalized spin-up and -down states $|s = \pm 1\rangle$, of the two-dimensional Hilbert space spanning the Clifford algebra of γ_i, which are eigenstates of the spin operator σ_3, $\sigma_3|s\rangle = s|s\rangle$. The full Hilbert space is obtained as the tensor product of these two representation spaces, leading to the orthonormalized basis $|n_+, n_-; s\rangle$ such that,

$$\langle n_+, n_-; s|m_+, m_-; s'\rangle = \delta_{n_+,m_+}\delta_{n_-,m_-}\delta_{s,s'}, \qquad \sum_{n_+,n_-=0}^{\infty}\sum_{s=\pm 1}|n_+, n_-; s\rangle\langle n_+, n_-; s| = \mathbb{I}. \quad (38)$$

These states diagonalize the Hamiltonian and angular-momentum operators already, but not yet the supercharge Q_0:

$$\begin{aligned} H|n_+, n_-; s\rangle &= \hbar\omega_c\left(n_- + \frac{1-s}{2}\right)|n_+, n_-; s\rangle, \\ L|n_+, n_-; s\rangle &= \hbar\left(n_+ - n_- + \frac{1}{2}s\right)|n_+, n_-; s\rangle. \end{aligned} \quad (39)$$

Note again the large degeneracy in the Landau levels, associated with the right-handed excitations of the bosonic sector, but extended further in the presence of the $\mathcal{N} = 1$ supersymmetry by a degeneracy in opposite spin values, $s = -1, +1$, for two adjacent bosonic Landau levels in $n_- = N - 1, N$, respectively. For any fixed value of $N = 1, 2, \ldots$ and whatever values $n_+, m_+ = 0, 1, 2, \ldots$, the states $|n_+, N-1; s = -1\rangle$ and $|m_+, N; s = +1\rangle$ are degenerate with energy $\hbar\omega_c N$.

This remark does not apply to the lowest energy sector, or lowest Landau level of the system, which consists of all the states $|n_+, 0; s = +1\rangle$ of vanishing energy ($n_+ = 0, 1, 2, \ldots$), and which are in fact supersymmetry invariant,

$$Q_0|n_+, 0; s = +1\rangle = 0. \quad (40)$$

The lowest Landau level thus provides a trivial representation of $\mathcal{N} = 1$ supersymmetry; each of its states being supersymmetry invariant. This also implies that supersymmetry remains unbroken by the quantum dynamics of the system.

For all the other Landau levels one has a nontrivial $\mathcal{N} = 1$ supersymmetry transformation and, as a matter of fact, a two-dimensional reducible representation for any given $n_+ = 0, 1, 2, \ldots$, and $N \geqslant 1$:

$$\begin{aligned} Q_0|n_+, N; s = +1\rangle &= \mathrm{i}\sqrt{\hbar\omega N}\,|n_+, N-1; s = -1\rangle, \\ Q_0|n_+, N-1; s = -1\rangle &= -\mathrm{i}\sqrt{\hbar\omega_c N}|n_+, N; s = +1\rangle. \end{aligned} \quad (41)$$

This result is obviously consistent with the property $Q_0^2 = H$ since these states each are already an eigenstate of H with energy $\hbar\omega_c N$. Furthermore, note they each are also an eigenstate of the angular-momentum operator L with common eigenvalue $\hbar(n_+ - N + 1/2)$. Consequently, for any given values for $n_+ = 0, 1, 2, \ldots$ and $N \geqslant 1$, one has the following two orthonormalized eigenstates of the $\mathcal{N} = 1$ supercharge Q_0:

$$|n_+, N; \delta\rangle \equiv \tfrac{1}{\sqrt{2}}[|n_+, N; s = +1\rangle + \mathrm{i}\delta|n_+, N-1; s = -1\rangle], \qquad \delta = \pm 1, \quad (42)$$

such that

$$Q_0|n_+, N; \delta\rangle = \delta\sqrt{\hbar\omega_c N}|n_+, N; \delta\rangle, \qquad N \geqslant 1, \quad \delta = \pm 1. \quad (43)$$

These eigenvalues are real since Q_0 is Hermitian, and correspond to the two square roots of the energy eigenvalue of that Landau level, $\hbar\omega_c N$.

In conclusion, an orthonormalized basis[10], which diagonalizes all three commuting operators H, L and Q_0, consists of the sets of states $|n_+, 0; s = +1\rangle$ and $|n_+, N; \delta\rangle$ with $n_+ = 0, 1, 2, \ldots, N = 1, 2, \ldots, \delta = \pm 1$ and the following resolution of the unit operator:

$$\mathbb{I} = \sum_{n_+=0}^{\infty} |n_+, 0; s = +1\rangle\langle n_+, 0; s = +1| + \sum_{\delta=\pm 1}\sum_{N=1}^{\infty}\sum_{n_+=0}^{\infty} |n_+, N; \delta\rangle\langle n_+, N; \delta|. \tag{44}$$

Furthermore,

$$\begin{aligned} &H|n_+, 0; s = +1\rangle = 0, \qquad L|n_+, 0; s = +1\rangle = \hbar\left(n_+ + \tfrac{1}{2}\right)|n_+, 0; s = +1\rangle, \\ &Q_0|n_+, 0; s = +1\rangle = 0, \end{aligned} \tag{45}$$

while for $N \geqslant 1$, $\delta = \pm 1$,

$$\begin{aligned} &H|n_+, N; \delta\rangle = \hbar\omega_c N|n_+, N; \delta\rangle, \qquad L|n_+, N; \delta\rangle = \hbar\left(n_+ - N + \tfrac{1}{2}\right)|n_+, N; \delta\rangle, \\ &Q_0|n_+, N; \delta\rangle = \delta\sqrt{\hbar\omega_c N}\,|n_+, N; \delta\rangle. \end{aligned} \tag{46}$$

For the discussion that follows, it is also useful to have available spectral decompositions of all elementary degrees of freedom in this basis. Let us introduce the following notation for the projection operators ($N \geqslant 1$):

$$\mathbb{P}_0 = \sum_{n_+=0}^{\infty} |n_+, 0; s = +1\rangle\langle n_+, 0; s = +1|, \qquad \mathbb{P}(N, \delta) = \sum_{n_+=0}^{\infty} |n_+, N; \delta\rangle\langle n_+, N; \delta|, \tag{47}$$

which are such that,

$$\mathbb{I} = \mathbb{P}_0 + \sum_{\delta=\pm 1}\sum_{N=1}^{\infty}\mathbb{P}(N, \delta), \quad H = \hbar\omega_c \sum_{\delta=\pm 1}\sum_{N=1}^{\infty} N\mathbb{P}(N, \delta), \quad Q_0 = \sqrt{\hbar\omega_c}\sum_{\delta=\pm 1}\sum_{N=1}^{\infty}\delta\mathbb{P}(N, \delta). \tag{48}$$

It is also convenient to introduce the following two quantities:

$$F_+(N) = \frac{\sqrt{N} + \sqrt{N-1}}{2}, \qquad F_-(N) = \frac{\sqrt{N} - \sqrt{N-1}}{2}, \qquad N \geqslant 1, \tag{49}$$

in terms of which one finds the following representations for the elementary degrees of freedom:

$$\begin{aligned} a_+ = &\sum_{n_+=0}^{\infty} |n_+, 0; s = +1\rangle\sqrt{n_+ + 1}\langle n_+ + 1, 0; s = +1| \\ &+ \sum_{\delta=\pm 1}\sum_{N=1}^{\infty}\sum_{n_+=0}^{\infty} |n_+, N; \delta\rangle\sqrt{n_+ + 1}\,\langle n_+ + 1, N; \delta|, \end{aligned} \tag{50}$$

$$\begin{aligned} a_+^\dagger = &\sum_{n_+=0}^{\infty} |n_+ + 1, 0; s = +1\rangle\sqrt{n_+ + 1}\langle n_+, 0; s = +1| \\ &+ \sum_{\delta=\pm 1}\sum_{N=1}^{\infty}\sum_{n_+=0}^{\infty} |n_+ + 1, N; \delta\rangle\sqrt{n_+ + 1}\,\langle n_+, N; \delta|, \end{aligned} \tag{51}$$

[10] Up to arbitrary phase redefinitions of each state, this basis is unique.

$$a_- = \sum_{\delta=\pm1}\sum_{n_+=0}^{\infty} |n_+, 0; s=+1\rangle \frac{1}{\sqrt{2}} \langle n_+, 1; \delta| + \sum_{\delta=\pm1}\sum_{N=1}^{\infty}\sum_{n_+=0}^{\infty} |n_+, N; \delta\rangle\, F_+(N) \langle n_+, N+1; \delta| + \sum_{\delta=\pm1}\sum_{N=1}^{\infty}\sum_{n_+=0}^{\infty} |n_+, N; \delta\rangle\, F_-(N) \langle n_+, N+1; -\delta|, \tag{52}$$

$$a_-^\dagger = \sum_{\delta=\pm1}\sum_{n_+=0}^{\infty} |n_+, 1; \delta\rangle \frac{1}{\sqrt{2}} \langle n_+, 0; s=+1| + \sum_{\delta=\pm1}\sum_{N=1}^{\infty}\sum_{n_+=0}^{\infty} |n_+, N+1; \delta\rangle\, F_+(N) \langle n_+, N; \delta| + \sum_{\delta=\pm1}\sum_{N=1}^{\infty}\sum_{n_+=0}^{\infty} |n_+, N+1; \delta\rangle\, F_-(N) \langle n_+, N; -\delta|, \tag{53}$$

and

$$\sigma_+ = \sum_{\delta=\pm1}\sum_{n_+=0}^{\infty} |n_+, 0; s=+1\rangle \frac{\mathrm{i}\delta}{\sqrt{2}} \langle n_+, 1; \delta| + \sum_{\delta,\delta'=\pm1}\sum_{N=1}^{\infty}\sum_{n_+=0}^{\infty} |n_+, N; \delta\rangle \frac{\mathrm{i}\delta'}{2} \langle n_+, N+1; \delta'|, \tag{54}$$

$$\sigma_- = \sum_{\delta=\pm1}\sum_{n_+=0}^{\infty} |n_+, 1; \delta\rangle \frac{-\mathrm{i}\delta}{\sqrt{2}} \langle n_+, 0; s=+1| + \sum_{\delta,\delta'=\pm1}\sum_{N=1}^{\infty}\sum_{n_+=0}^{\infty} |n_+, N+1; \delta\rangle \frac{-\mathrm{i}\delta}{2} \langle n_+, N; \delta'|. \tag{55}$$

3. Supersymmetric Landau level projections

In the same spirit that led to the noncommutative Moyal–Voros plane by projection of the ordinary $\mathcal{N}=0$ Landau problem onto any of its Landau levels, we would now like to consider similar projections which preserve the $\mathcal{N}=1$ supersymmetry in order to identify an $\mathcal{N}=1$ supersymmetric extended non(anti)commutative Moyal–Voros superplane.

Clearly the supersymmetric invariant lowest Landau level $|n_+, 0; s=+1\rangle$ is of no use in that respect. A projection onto that single level produces the ordinary $\mathcal{N}=0$ noncommutative Moyal–Voros plane of section 2. Likewise, as may also be seen from the above expressions, projecting onto a single supersymmetric covariant Landau level $|n_+, N; \delta\rangle$ ($N \geqslant 1$), using $\mathbb{P}(N, \delta)$, once again results in the projection of the spin degrees of freedom, $\sigma_\pm$, and the left-handed chiral bosonic modes, $\left(a_-, a_-^\dagger\right)$ onto the null operator, leaving simply the bosonic noncommutative Moyal–Voros plane of section 2 spanned by $\left(a_+, a_+^\dagger\right)$. One thus has to resort to more than a single Landau sector $|n_+, N; \delta\rangle$ in order to gain something new.

Restricting to a single energy eigenvalue, $\hbar\omega_c N$ with $N \geqslant 1$ say, a natural choice appears to be given by a projector combining the two supersymmetry eigenvalues with $\delta = \pm1$:

$$\mathbb{P}(N) = \mathbb{P}(N, \delta=+1) + \mathbb{P}(N, \delta=-1). \tag{56}$$

This projection is such that (again in the notation that $\overline{A} = \mathbb{P}(N)A\,\mathbb{P}(N)$ for some operator A),

$$\overline{\sigma}_+ = 0, \qquad \overline{\sigma}_- = 0, \qquad \overline{a}_- = 0, \qquad \overline{a_-^\dagger} = \overline{a}_-^\dagger = 0, \tag{57}$$

while

$$\begin{aligned} \overline{a}_+ &= \sum_{\delta=\pm1}\sum_{n_+=0}^{\infty} |n_+, N; \delta\rangle\sqrt{n_+ + 1}\langle n_+ + 1, N; \delta|, \\ \overline{a_+^\dagger} &= \overline{a}_+^\dagger = \sum_{\delta=\pm1}\sum_{n_+=0}^{\infty} |n_+ + 1, N; \delta\rangle\sqrt{n_+ + 1}\langle n_+, N; \delta|, \end{aligned} \tag{58}$$

as well as

$$\begin{aligned} &\overline{\sigma}_3 = \mathbb{P}(N), \qquad \overline{H} = \hbar\omega_c N\mathbb{P}(N), \qquad \overline{Q}_0 = \sqrt{\hbar\omega_c}[\mathbb{P}(N,+1) - \mathbb{P}(N,-1)], \\ &\overline{L} = \hbar\sum_{\delta=\pm1}\sum_{n_+=0}^{\infty} |n_+, N; \delta\rangle\left(n_+ - N + \frac{1}{2}\right)\langle n_+, N; \delta|. \end{aligned} \tag{59}$$

Hence, using the projector $\mathbb{P}(N)$ one in fact simply recovers two commuting copies of the $\mathcal{N} = 0$ noncommutative Moyal–Voros plane, distinguished by the sign $\delta = \pm1$ of the Q_0 eigenvalues at Landau level N. Both the projected left-handed chiral bosonic modes, $\left(a_-, a_-^\dagger\right)$ (as in section 2), and the fermionic modes, $\sigma_\pm$, are reduced to the null operator, leaving only the Fock algebra of the projected right-handed chiral bosonic modes:

$$\left[\overline{a}_+, \overline{a}_+^\dagger\right] = \mathbb{P}(N). \tag{60}$$

Consequently, in order to gain some new structure, one needs to consider a projection involving at least two[11] Landau sectors separated by the quantum energy gap $\hbar\omega_c$.

3.1. The two Landau level projection

Given a pair of fixed values $\delta = \pm1$ and $N \geqslant 1$, let us consider the projection associated with the projector $\mathbb{P} = \mathbb{P}_+$ defined as, together with its counterpart $\mathbb{P}_-$,

$$\mathbb{P} = \mathbb{P}_+ = \mathbb{P}(N,\delta) + \mathbb{P}(N+1,\epsilon\delta), \qquad \mathbb{P}_- = \mathbb{P}(N,\delta) - \mathbb{P}(N+1,\epsilon\delta), \tag{61}$$

where $\epsilon = \pm1$. In this manner the projection onto the two energy sectors at levels N and $(N+1)$ involves either the same or opposite signs for the supercharge eigenvalues. Including all possibilities $\delta = \pm1$ and $\epsilon = \pm1$ accounts for all four such combinations given the levels N and $(N+1)$.

This projection is such that,

$$\overline{\sigma}_3 = 0, \tag{62}$$

while for the fermionic degrees of freedom,

$$\overline{\sigma}_+ = \sum_{n_+=0}^{\infty} |n_+, N; \delta\rangle\,\frac{\mathrm{i}\epsilon\delta}{2}\langle n_+, N+1; \epsilon\delta|, \qquad \overline{\sigma}_- = \sum_{n_+=0}^{\infty} |n_+, N+1; \epsilon\delta\rangle\,\frac{-\mathrm{i}\epsilon\delta}{2}\langle n_+, N; \delta|, \tag{63}$$

which are such that,

$$\overline{\sigma}_+^\dagger = \overline{\sigma}_-, \qquad \overline{\sigma}_-^\dagger = \overline{\sigma}_+. \tag{64}$$

[11] Like in the $\mathcal{N} = 0$ case, including more than two energy levels does not seem to offer any particular advantages.

The projected bosonic degrees of freedom are given as

$$\begin{aligned}\overline{a}_- &= \sum_{n_+=0}^{\infty} |n_+, N; \delta\rangle\, F_\epsilon(N) \langle n_+, N+1; \epsilon\delta|,\\ \overline{a_-^\dagger} &= \overline{a}_-^\dagger = \sum_{n_+=0}^{\infty} |n_+, N+1; \epsilon\delta\rangle\, F_\epsilon(N) \langle n_+, N; \delta|,\end{aligned} \tag{65}$$

and

$$\begin{aligned}\overline{a}_+ &= \sum_{n_+=0}^{\infty} |n_+, N; \delta\rangle \sqrt{n_+ + 1} \langle n_+ + 1, N; \delta|\\ &\quad + \sum_{n_+=0}^{\infty} |n_+, N+1; \epsilon\delta\rangle \sqrt{n_+ + 1} \langle n_+ + 1, N+1; \epsilon\delta|,\\ \overline{a_+^\dagger} &= \overline{a}_+^\dagger = \sum_{n_+=0}^{\infty} |n_+ + 1, N; \delta\rangle \sqrt{n_+ + 1} \langle n_+, N; \delta|\\ &\quad + \sum_{n_+=0}^{\infty} |n_+ + 1, N+1; \epsilon\delta\rangle \sqrt{n_+ + 1} \langle n_+, N+1; \epsilon\delta|.\end{aligned} \tag{66}$$

For the remaining operators of interest, one has

$$\begin{aligned}\overline{H} &= \hbar\omega_c [N\mathbb{P}(N, \delta) + (N+1)\mathbb{P}(N+1, \epsilon\delta)] = \overline{Q}_0^2,\\ \overline{Q}_0 &= \delta\sqrt{\hbar\omega_c}[\sqrt{N}\mathbb{P}(N, \delta) + \epsilon\sqrt{N+1}\mathbb{P}(N+1, \epsilon\delta)],\\ \overline{L} &= \hbar\left[\overline{a}_+^\dagger \overline{a}_+ - \left(N - \tfrac{1}{2}\right)\mathbb{P}(N, \delta) - \left(N + \tfrac{1}{2}\right)\mathbb{P}(N+1, \epsilon\delta)\right].\end{aligned} \tag{67}$$

Hence, this projection certainly leads to some new algebraic structure consistent with the $\mathcal{N} = 1$ supersymmetry since we still have $\overline{Q}_0^2 = \overline{H}$, while the projected left-handed chiral bosonic as well as the fermionic contents remain nontrivial. Yet, like in the $\mathcal{N} = 0$ case the number of projected phase space degrees of freedom is reduced by two since the latter two sets of degrees of freedom are no longer independent. Indeed, a direct inspection of the above expressions shows that one has

$$\overline{a}_- + 2\mathrm{i}\epsilon\delta F_\epsilon(N)\overline{\sigma}_+ = 0, \qquad \overline{a}_-^\dagger - 2\mathrm{i}\epsilon\delta F_\epsilon(N)\overline{\sigma}_- = 0. \tag{68}$$

In other words, rather than the two combinations $(p_1 + Bx_2/2)$ and $(p_2 - Bx_1/2)$, which upon projection vanish in the $\mathcal{N} = 0$ case[12], the two combinations of degrees of freedom that vanish given the present projection in the $\mathcal{N} = 1$ case are those above.

This projection thus effects a boson–fermion transmutation of sorts! Bosonic degrees of freedom $\overline{a}_-^{(\dagger)}$ obey specific commutation relations, while fermionic degrees of freedom $\overline{\sigma}_\pm$ obey specific anticommutation relations, which in each case specify precisely this spin-statistics property. However, through the considered projection by $\mathbb{P} = \mathbb{P}_+$, these two sets of degrees of freedom become identified and coalesce into one another, and hence are characterized by both commutation and anticommutation relations which are independent of, but consistent with one another. The considered $\mathcal{N} = 1$ supersymmetric invariant projection results in two bosonic and two fermionic phase space variables to become identified, and yet preserving both these statistics properties. Since this proves to be convenient, hereafter we express quantities in terms of $\overline{\sigma}_\pm$ only, knowing that at the same time these variables stand for $\overline{a}_-^{(\dagger)}$ as well. Given this dual rôle, the variables $\overline{\sigma}_\pm$ are thus characterized by both commutation and

[12] These two combinations do not vanish for the present $\mathcal{N} = 1$ supersymmetry covariant projection.

anticommutation properties. For instance, besides those operators already mentioned above, the projected supercharge $\overline{Q}_0$ also possesses well-defined commutation and anticommutation relations with these bosonic–fermionic variables. This is quite an intriguing outcome of the effected projection.

3.2. The algebra of the $\mathcal{N} = 1$ supersymmetric Moyal–Voros superplane

Since the commutators and anticommutators of $\overline{\sigma}_\pm$ are now both specified, so are their bilinear products. One finds

$$\overline{\sigma}_+^2 = 0, \qquad \overline{\sigma}_-^2 = 0, \qquad \overline{\sigma}_+\overline{\sigma}_- = \tfrac{1}{4}\mathbb{P}(N,\delta), \qquad \overline{\sigma}_-\overline{\sigma}_+ = \tfrac{1}{4}\mathbb{P}(N+1,\epsilon\delta). \tag{69}$$

Introducing the operators

$$\overline{\sigma}_1 = \overline{\sigma}_+ + \overline{\sigma}_-, \qquad \overline{\sigma}_2 = -\mathrm{i}\,(\overline{\sigma}_+ - \overline{\sigma}_-)\,, \tag{70}$$

one has, likewise,

$$\overline{\sigma}_1^2 = \tfrac{1}{4}\mathbb{P}_+, \qquad \overline{\sigma}_2^2 = \tfrac{1}{4}\mathbb{P}_+, \qquad \overline{\sigma}_1\overline{\sigma}_2 = \tfrac{1}{4}\mathrm{i}\mathbb{P}_-, \qquad \overline{\sigma}_2\overline{\sigma}_1 = -\tfrac{1}{4}\mathrm{i}\mathbb{P}_-. \tag{71}$$

Consequently, the full algebra of commutation and anticommutation relations for these bosonic–fermionic variables is given as

$$\overline{\sigma}_+^2 = 0, \qquad \overline{\sigma}_-^2 = 0, \qquad \{\overline{\sigma}_+,\overline{\sigma}_-\} = \tfrac{1}{4}\mathbb{P}_+, \qquad [\overline{\sigma}_+,\overline{\sigma}_-] = \tfrac{1}{4}\mathbb{P}_-, \tag{72}$$

$$\{\overline{\sigma}_i,\overline{\sigma}_j\} = \tfrac{1}{4}\,2\delta_{ij}\,\mathbb{P}_+, \qquad [\overline{\sigma}_i,\overline{\sigma}_j] = \tfrac{1}{2}\mathrm{i}\epsilon_{ij}\,\mathbb{P}_-. \tag{73}$$

Note the consistency of these expressions with the manifest $SO(2)$ covariance properties under rotations in the plane.

Besides these two bosonic–fermionic variables, the $\mathcal{N} = 1$ Moyal–Voros superplane also consists of the projected right-handed chiral bosonic variables $a_+^{(\dagger)}$, making up a total of four variables for that non(anti)commutative variety. This sector commutes with the previous one:

$$[\overline{a}_+,\overline{\sigma}_\pm] = 0, \qquad \left[\overline{a}_+^\dagger,\overline{\sigma}_\pm\right] = 0, \qquad [\overline{a}_+,\overline{\sigma}_i] = 0, \qquad \left[\overline{a}_+^\dagger,\overline{\sigma}_i\right] = 0, \tag{74}$$

while these two operators also define a Fock algebra on the projected space:

$$\left[\overline{a}_+,\overline{a}_+^\dagger\right] = \mathbb{P}_+. \tag{75}$$

In effect this sector alone consists of two copies of the $\mathcal{N} = 0$ Moyal–Voros plane, soldered with the remaining sector of bosonic–fermionic variables $\overline{\sigma}_\pm$, to build up the non(anti)commutative space of the projected original Cartesian coordinates of the Euclidean plane, as we now proceed to show.

One finds for the projected Cartesian coordinates

$$\overline{x}_1 = \sqrt{\frac{\hbar}{2B}}\left[\left(\overline{a}_+ + \overline{a}_+^\dagger\right) + 2\epsilon\delta\,F_\epsilon(N)\,\overline{\sigma}_2\right], \qquad \overline{x}_2 = \sqrt{\frac{\hbar}{2B}}\left[\mathrm{i}\left(\overline{a}_+ - \overline{a}_+^\dagger\right) - 2\epsilon\delta\,F_\epsilon(N)\,\overline{\sigma}_1\right], \tag{76}$$

while the projected conjugate momenta are such that,

$$\overline{p}_i + \tfrac{1}{2}B\epsilon_{ij}\overline{x}_j = -\epsilon\delta\sqrt{2\hbar B}\,F_\epsilon(N)\,\overline{\sigma}_i. \tag{77}$$

Hence, rather than vanish as in the $\mathcal{N} = 0$ case, these specific combinations of projected bosonic operators reduce purely to the bosonic–fermionic variables $\overline{\sigma}_i$. Thus, from this point of view one may consider the projected Cartesian coordinates, $\overline{x}_i$, together with the bosonic–fermionic spin variables, $\overline{\sigma}_i$, to provide the complete set of independent variables spanning

the $\mathcal{N} = 1$ supersymmetric Moyal–Voros superplane. Consequently, the only independent commutation relations still to be considered are those among these quantities. One finds

$$[\overline{x}_i, \overline{x}_j] = -\frac{\mathrm{i}\hbar}{B}\epsilon_{ij}\big(\mathbb{P}_+ - F_\epsilon^2(N)\,\mathbb{P}_-\big), \tag{78}$$

which thus specifies the noncommutative geometry of the superplane in its bosonic sector, while

$$[\overline{x}_i, \overline{\sigma}_j] = -\mathrm{i}\epsilon\delta\sqrt{\frac{\hbar}{2B}}\,F_\epsilon(N)\delta_{ij}\mathbb{P}_- \tag{79}$$

specifies that noncommutativity between the bosonic and fermionic sectors, while the remaining structures of (anti)commutators in (73) in the fermionic sector characterize the mixed bosonic–fermionic character of the latter. Once again note the manifest $SO(2)$ covariance properties of all these relations for rotations in the plane.

3.3. Covariance properties of $\mathcal{N} = 1$ supersymmetric Moyal–Voros superplanes

Besides the algebraic relations characterizing the $\mathcal{N} = 1$ supersymmetric Moyal–Voros superplane, it is also of interest, because of its built-in supersymmetric covariance properties, to consider the action of the projected $\mathcal{N} = 1$ supercharge on the defining variables of the superplane. In order to present the results of later interest and of somewhat more general use, let us consider a slight generalization of the operator $\overline{Q}_0$ in the form

$$Q_1 = \sqrt{\hbar\omega_c}(\alpha_+\mathbb{P}(N,\delta) + \alpha_-\mathbb{P}(N+1,\epsilon\delta)), \tag{80}$$

such that

$$Q_1^2 = \hbar\omega_c\big(\alpha_+^2\mathbb{P}(N,\delta) + \alpha_-^2\,\mathbb{P}(N+1,\epsilon\delta)\big). \tag{81}$$

With the choice $\alpha_+ = \delta\sqrt{N}$ and $\alpha_- = \epsilon\delta\sqrt{N+1}$, Q_1 corresponds to the operator $\overline{Q}_0$.

Since the right-handed chiral bosonic sector is $\mathcal{N} = 1$ supersymmetric invariant, it follows that

$$\big[Q_1, \overline{a}_+\big] = 0, \qquad \big[Q_1, \overline{a}_+^\dagger\big] = 0. \tag{82}$$

Once again both the commutation and anticommutation relations of the variables $\overline{\sigma}_\pm$ with the supercharge are specified by the structure of the projection, and their products with Q_1 are easily identified:

$$Q_1\overline{\sigma}_\pm = \sqrt{\hbar\omega_c}\,\alpha_\pm\overline{\sigma}_\pm, \qquad \overline{\sigma}_\pm Q_1 = \sqrt{\hbar\omega_c}\,\alpha_\mp\overline{\sigma}_\pm. \tag{83}$$

Similar relations hold for the products $Q_1\overline{\sigma}_i$ and $\overline{\sigma}_i Q_1$, which can easily be worked out from these. Consequently,

$$[Q_1, \overline{\sigma}_\pm] = \pm\sqrt{\hbar\omega_c}(\alpha_+ - \alpha_-)\overline{\sigma}_\pm, \qquad \{Q_1, \overline{\sigma}_\pm\} = \sqrt{\hbar\omega_c}(\alpha_+ + \alpha_-)\overline{\sigma}_\pm, \tag{84}$$

as well as

$$[Q_1, \overline{\sigma}_i] = \mathrm{i}\sqrt{\hbar\omega_c}(\alpha_+ - \alpha_-)\epsilon_{ij}\overline{\sigma}_j, \qquad \big\{Q_1, \overline{\sigma}_i\big\} = \sqrt{\hbar\omega_c}\,(\alpha_+ + \alpha_-)\,\overline{\sigma}_i. \tag{85}$$

It then also follows that for the Cartesian superplane coordinates,

$$[Q_1, \overline{x}_i] = -\frac{\mathrm{i}\hbar}{\sqrt{2m}}\,2\epsilon\delta\,F_\epsilon(N)(\alpha_+ - \alpha_-)\,\overline{\sigma}_i, \tag{86}$$

showing that indeed under the $\mathcal{N} = 1$ supersymmetry transformations these variables are mapped into the spin degrees of freedom, while the latter, due to their dual bosonic–fermionic character, are simply mapped back into themselves, in a manner still consistent with supersymmetry.

Under $SO(2)$ planar rotations, the projected variables of the Moyal–Voros superplane transform according to the relations

$$\left[\overline{L}, \overline{a}_+^\dagger\right] = \hbar \overline{a}_+^\dagger, \qquad [\overline{L}, \overline{a}_+] = -\hbar \overline{a}_+, \qquad [\overline{L}, \overline{\sigma}_\pm] = \pm\hbar \overline{\sigma}_\pm, \qquad [\overline{L}, \overline{\sigma}_i] = \mathrm{i}\hbar \epsilon_{ij} \overline{\sigma}_j. \tag{87}$$

These properties are identical to the corresponding commutation relations before projection (because L commutes with the considered projection). Hence, indeed, the applied projection remains manifestly covariant under both $\mathcal{N} = 1$ supersymmetry and $SO(2)$ rotations in the Euclidean plane.

4. Conclusions

By considering the $\mathcal{N} = 1$ supersymmetric Landau problem and nontrivial projections onto its Landau levels consistent with supersymmetry, a $\mathcal{N} = 1$ supersymmetric covariant analogue of the ordinary noncommutative Moyal–Voros plane has been identified, independently of the choice of signs (δ, ϵ) and involving some normalization factors dependent on the choice of Landau levels N and $(N + 1)$ onto which the projection is performed. To perhaps better make manifest the general structure that has thereby emerged, it is useful to apply the following changes of notation:

$$\begin{aligned} &|n_+, N; \delta\rangle \to |n; \tau = +1\rangle, && |n_+, N+1; \epsilon\delta\rangle \to |n; \tau = -1\rangle, \\ &\mathbb{P}(N, \delta) \to \mathbb{P}(\tau = +1), && \mathbb{P}(N+1, \epsilon\delta) \to \mathbb{P}(\tau = -1), \\ &\mathbb{P}_+ \to \mathbb{E} = \mathbb{P}(+1) + \mathbb{P}(-1), && \mathbb{P}_- \to \tau_3 = \mathbb{P}(+1) - \mathbb{P}(-1). \end{aligned} \tag{88}$$

In this notation

$$\begin{aligned} &\overline{a}_+ \to b = \sum_{\tau=\pm1} \sum_{n=0}^{\infty} |n; \tau\rangle \sqrt{n+1} \langle n+1; \tau|, && \overline{a}_+^\dagger \to b^\dagger = \sum_{\tau\pm1} \sum_{n=0}^{\infty} |n+1; \tau\rangle \sqrt{n+1} \langle n; \tau|, \\ &-2\mathrm{i}\epsilon\delta\overline{\sigma}_+ \to \tau_+ = \sum_{n=0}^{\infty} |n; +1\rangle\langle n; -1|, && +2\mathrm{i}\epsilon\delta\overline{\sigma}_- \to \tau_- = \sum_{n=0}^{\infty} |n; -1\rangle\langle n; +1|. \end{aligned} \tag{89}$$

It is also useful to define the following combinations:

$$\tau_1 = \mathrm{i}\,(\tau_+ - \tau_-) = 2\epsilon\delta\overline{\sigma}_1, \qquad \tau_2 = \tau_+ + \tau_- = 2\epsilon\delta\overline{\sigma}_2. \tag{90}$$

Instead of the $(b, b^\dagger)$ Fock operators, let us rather introduce a Cartesian $\mathcal{N} = 0$ Moyal–Voros plane parametrization in the form

$$u_1 = \sqrt{\frac{\hbar}{2B}}(b + b^\dagger), \qquad u_2 = \sqrt{\frac{\hbar}{2B}}\,\mathrm{i}(b - b^\dagger). \tag{91}$$

Finally, let us introduce the real dimensionless parameter λ in place of the quantity $F_\epsilon(N)$.

In terms of this new notation, the structure of the $\mathcal{N} = 1$ non(anti)commutative Moyal–Voros superplane is as follows. It is spanned by the operators $\overline{x}_i$ and τ_i $(i = 1, 2)$, $\overline{x}_i$ being the Cartesian bosonic variables and τ_i being bosonic–fermionic spin degrees of freedom, with

the parametrization[13]

$$\overline{x}_i = u_i + \lambda\sqrt{\frac{\hbar}{2B}}\epsilon_{ij}\,\tau_j. \tag{92}$$

The algebra of these operators is characterized by the following (anti)commutation relations:

$$[u_i, u_j] = -\frac{\mathrm{i}\hbar}{B}\,\mathbb{E}, \tag{93}$$

$$\tau_+^2 = 0, \qquad \tau_-^2 = 0, \qquad \{\tau_+, \tau_-\} = \mathbb{E}, \qquad \{\tau_i, \tau_j\} = 2\delta_{ij}\,\mathbb{E}, \tag{94}$$

$$[\tau_+, \tau_-] = \tau_3, \qquad [\tau_i, \tau_j] = 2\mathrm{i}\epsilon_{ij}\,\tau_3. \tag{95}$$

Consequently,

$$[\overline{x}_i, \overline{x}_j] = -\frac{\mathrm{i}\hbar}{B}\epsilon_{ij}(\mathbb{E} - \lambda^2\,\tau_3), \qquad [\overline{x}_i, \tau_j] = -2\mathrm{i}\lambda\sqrt{\frac{\hbar}{2B}}\,\delta_{ij}\,\tau_3. \tag{96}$$

Note that the bosonic–fermionic spin degrees of freedom sector is such that the only possible representation of both the anticommutation relations and $SU(2)$ commutation relations of the Clifford algebra is in terms of the 2×2 Pauli matrices, in the case of the present $\mathcal{N} = 1$ supersymmetry construction. What is perhaps even more intriguing is that the bosonic sector of the superplane coordinates, $\overline{x}_i$, is realized through a soldering with the spin degrees of freedom τ_i of two copies—distinguished by the eigenvalues $\tau = \pm 1$ of τ_3—of the ordinary $\mathcal{N} = 0$ noncommutative Moyal–Voros plane spanned by u_i, while the free parameter λ sets the strength of that soldering. It is as if the ordinary Moyal–Voros plane had been 'fattened' by soldering onto it a spin-1/2 structure, while at the same time making the notion of a bosonic or a fermionic variable more fuzzy since τ_i possess both characters while $\overline{x}_i$ is constructed as a linear combination out of these as well as a bosonic Fock algebra.

Considering a supersymmetry-like charge,

$$Q_1 = \sqrt{\hbar\omega_c}[\alpha_+\mathbb{P}(+1) + \alpha_-\mathbb{P}(-)], \tag{97}$$

$\alpha_\pm$ being two real arbitrary constants, which leaves all states $|n;\tau\rangle$ invariant, and for which we thus have

$$Q_1^2 = \hbar\omega_c\big[\alpha_+^2\mathbb{P}(+1) + \alpha_-^2\mathbb{P}(-)\big]; \tag{98}$$

its action on the superplane supercoordinates is such that,

$$[Q_1, u_i] = 0, \qquad [Q_1, \overline{x}_i] = -\mathrm{i}\frac{\hbar}{\sqrt{2m}}\,\lambda(\alpha_+ - \alpha_-)\,\tau_i, \tag{99}$$

and

$$[Q_1, \tau_\pm] = \pm\sqrt{\hbar\omega_c}(\alpha_+ - \alpha_-)\,\tau_\pm, \qquad \{Q_1, \tau_\pm\} = \sqrt{\hbar\omega_c}\,(\alpha_+ + \alpha_-)\,\tau_\pm, \tag{100}$$

$$[Q_1, \tau_i] = \mathrm{i}\sqrt{\hbar\omega_c}(\alpha_+ - \alpha_-)\epsilon_{ij}\tau_j, \qquad \{Q_1, \tau_i\} = \sqrt{\hbar\omega_c}\,(\alpha_+ + \alpha_-)\,\tau_i. \tag{101}$$

In the Landau problem context one has $\alpha_+ = \delta\sqrt{N}$ and $\alpha_- = \epsilon\delta\sqrt{N+1}$ with $\lambda = F_\epsilon(N)$ $(N \geqslant 1)$. However, in a more general context, given only the two-sector structure of the representation space of the $\mathcal{N} = 1$ Moyal–Voros superplane in the quantum number $\tau = \pm 1$, one is free to choose these three real parameters, with in particular Q_1^2 then playing the role of a Hamiltonian operator for the modelling of some physical system. For instance by choosing $\alpha_+ = \alpha_- = \alpha$, the two sectors, $\tau = \pm 1$, are degenerate in energy and $SO(2)$ covariant

[13] Note that $\sqrt{\hbar/(2B)}$ is the magnetic length of the Landau problem.

with states which are all eigenstates of Q_1 with a common eigenvalue $\alpha\sqrt{\hbar\omega_c}$, while all supercoordinates $\overline{x}_i$ and τ_i have vanishing commutators with Q_1.

However, it remains to be seen whether such a $\mathcal{N} = 1$ Moyal–Voros superplane framework is of any relevance to the quantum Hall problem, be it in its integer or fractional realizations. Furthermore, the above picture of two ordinary Moyal–Voros planes soldered by the bosonic–fermionic spin degrees of freedom is strangely reminiscent of two D-branes stacked on top of one another [16] in a limit leading to noncommutativity in M-theory [7]. It may be worth understanding the possible relation between these two situations, if any.

Besides such wider ranging physical issues, given the present construction, one interesting task remaining to be completed now is the identification of a Grassmann graded $\star$-product defined on the functions of an $\mathcal{N} = 1$ supersymmetric extension of the commutative Cartesian coordinates, (x_1, x_2), of the plane [6]. Beyond that, the generalization of the present approach by including a larger number of supersymmetries is certainly worth considering in order to enrich the collection of such Moyal–Voros superplanes beyond the $\mathcal{N} = 1$ case [13].

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